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Convergence and asymptotic behavior of Jackson integrals associated with irreducible reduced root systems

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Abstract

For a multivariable q-series called Jackson integral associated with irreducible reduced root systems, a sufficient condition for convergence of it with respect to parameters is given. Its asymptotic behavior as a function of its parameters is studied. For its application, we give another proof of G_2-type summation formula investigated by Gustafson in an appendix.

1. Introduction

In the previous paper [8], we defined the Jackson integral associated with irreducible reduced root system. It is a natural multivariable extension of both the Ramanujan's ${}_1\psi_1$ sum

$$\sum_{\nu=-\infty}^{\infty} \frac{(a)_\nu}{(b)_\nu} z^\nu = \frac{(az)_\infty}{(z)_\infty} \frac{(q)_\infty}{(b)_\infty} \frac{(b/a)_\infty}{(q/a)_\infty} \frac{(q/az)_\infty}{(b/az)_\infty}$$

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and Bailey's very-well-poised ${}_6\psi_6$ sum

$$\sum_{v=-\infty}^{\infty} \frac{(1-aq^{2v})(b)_v(c)_v(d)_v(e)_v}{(1-a)(aq/b)_v(aq/c)_v(aq/d)_v(aq/e)_v}\left(\frac{a^2q}{bcde}\right)^v$$
$$= \frac{(q/a)_\infty(aq)_\infty(aq/bc)_\infty(aq/bd)_\infty(aq/be)_\infty}{(q/b)_\infty(q/c)_\infty(q/d)_\infty(q/e)_\infty(aq/b)_\infty}$$
$$\times \frac{(aq/cd)_\infty(aq/ce)_\infty(aq/de)_\infty(q)_\infty}{(aq/c)_\infty(aq/d)_\infty(aq/e)_\infty(a^2q/bcde)_\infty}.$$

Gustafson [4] established multidimensional q-series generalization of Bailey's ${}_6\psi_6$ summation formula corresponding to simple Lie algebras. On the other hand, Aomoto [1] extended q-Selberg integral to a sum (q-series) which has a symmetry of Weyl group of irreducible reduced root system. Gustafson's sums and Aomoto's sums are very similar. Indeed, By using Gustafson's C_n-type sum, van Diejen [11] proved a summation formula for his BC_n-type sum, which includes Aomoto's B_n and C_n-type sums as special cases. One of motivations of considering our Jackson integrals, which include most of their sums, is to treat their summation formulas together. Their multivariable q-series can be expressed as a product of q-gamma function and a Jacobi elliptic theta function. We discussed in [8] when the Jackson integral can be expressed as a product of the theta functions. See Proposition 1 including a summation formula for F_4-type [9] which seems to be new. See also Theorem A.3.

But first of all, in order to carry out the program outlined above, the Jackson integral under consideration should converge. Since it is an infinite sum over a lattice, we fail to define it if it diverges. We give a sufficient condition for its convergence with respect to parameters. See Theorem 4 in Section 3. This is one of the main results of this paper. It also assures the convergence of a q-series, which we call the Macdonald type sum in this paper, essentially introduced by Macdonald [10], who showed the relation between Aomoto's sum and the q-Macdonald–Morris identity investigated by Cherednik [2] and many others. Technically speaking, as we shall see in Appendix later, for evaluation of Jackson integral, our method needs repeated use of its difference equation with respect to parameters and its asymptotic behavior. In this process, however, it is important to keep the parameters within the convergence region of Jackson integral when we take parameters away to infinity, even if the sum is well-defined in the region. We have to choose a good direction of parameter shift. We show its asymptotic formula (see Theorem 6 in Section 4), which is another main result of this paper. This formula is interesting by itself. See also Proposition 5. As an example of its applications, we give another proof of Gustafson's G_2-type summation formula in Appendix.

Throughout this paper, we assume $0<q<1$ and use notation $(a)_\infty := \prod_{i=0}^{\infty}(1-aq^i)$ and $(a)_v = (a)_\infty/(aq^v)_\infty$.

2. Definition of Jackson integral

Let R be an irreducible reduced root system, spanning a real vector space E of dimension n, and let $\langle\cdot,\cdot\rangle$ be a positive definite scalar product on E invariant under the Weyl group W of R. We denote by R^+ the set of positive roots relative to a fixed basis $\{\alpha_1,\dots,\alpha_n\}$ of R. For each $\alpha\in R$, let $\alpha^\vee=2\alpha/\langle\alpha,\alpha\rangle$. Let P be the *coweight lattice* $\{\chi\in E;\ \langle\alpha,\chi\rangle\in\mathbf{Z}$ for any $\alpha\in R\}$ and let Q be the *coroot lattice* of R defined by $Q=\mathbf{Z}\alpha_1^\vee+\cdots+\mathbf{Z}\alpha_n^\vee\subset P$. Let L be any sublattice of P of rank n. We assume L is W-stable, i.e., $L=wL$ for $w\in W$. The scalar product $\langle\cdot,\cdot\rangle$ is uniquely extended linearly to $E_{\mathbf{C}}=E\otimes_{\mathbf{R}}\mathbf{C}\simeq\mathbf{C}^n$. For $x\in E_{\mathbf{C}}$, we define

$$\begin{aligned}\Phi_R(b_1,\dots,b_s,c_1,\dots,c_l;x)&=\Phi_R(\{b_i\},\{c_j\};x)\\&:=\prod_{i=1}^{s}\prod_{\substack{\alpha>0\\ \alpha:\text{short}}}q^{(\frac12-b_i)\langle\alpha,x\rangle}\frac{(q^{1-b_i+\langle\alpha,x\rangle})_\infty}{(q^{b_i+\langle\alpha,x\rangle})_\infty}\\&\quad\times\prod_{j=1}^{l}\prod_{\substack{\alpha>0\\ \alpha:\text{long}}}q^{(\frac12-c_j)\langle\alpha,x\rangle}\frac{(q^{1-c_j+\langle\alpha,x\rangle})_\infty}{(q^{c_j+\langle\alpha,x\rangle})_\infty},\end{aligned}\tag{1}$$

where $s,l\in\mathbf{Z}_{\geqslant0}$, $b_i,c_j\in\mathbf{C}$ and $\alpha>0$ means $\alpha\in R^+$. If all roots $\alpha\in R$ have the same length, we regard the roots as all short. We denote by $\Delta_R(x)$ the Weyl denominator as

$$\Delta_R(x):=\prod_{\alpha>0}(q^{\frac12\langle\alpha,x\rangle}-q^{-\frac12\langle\alpha,x\rangle}).\tag{2}$$

Let $U_w(x)$ be a function defined by

$$\begin{aligned}U_w(x):=&\prod_{i=1}^{s}\prod_{\substack{\alpha>0\\ -w^{-1}\alpha>0\\ \alpha:\text{short}}}q^{(2b_i-1)\langle\alpha,x\rangle}\frac{\theta(q^{b_i+\langle\alpha,x\rangle})}{\theta(q^{1-b_i+\langle\alpha,x\rangle})}\\&\times\prod_{j=1}^{l}\prod_{\substack{\alpha>0\\ -w^{-1}\alpha>0\\ \alpha:\text{long}}}q^{(2c_j-1)\langle\alpha,x\rangle}\frac{\theta(q^{c_j+\langle\alpha,x\rangle})}{\theta(q^{1-c_j+\langle\alpha,x\rangle})},\end{aligned}$$

where $\theta(\xi):=(\xi)_\infty(q/\xi)_\infty$. The function $\theta(\xi)$ has the *quasi-periodicity*

$$\theta(q\xi)=-\theta(\xi)/\xi.\tag{3}$$

This gives the following formula which shall be used in Section 4:

$$\theta(q^N\xi)=(-1)^N\xi^{-N}q^{-N(N-1)/2}\theta(\xi).\tag{4}$$

From (3), we see the function $U_w(x)$ is a *pseudo-constant*, i.e., an invariant under the shift $x\to x+\chi$ for $\chi\in P$.

For $w \in W$, we define $wF(x) := F(w^{-1}x)$ for a function $F(x)$ of $x \in E_{\mathbf{C}}$. The function $\Phi_R(\{b_i\},\{c_j\};x)$ is *quasi W-symmetric* with respect to W:

$$w\Phi_R(\{b_i\},\{c_j\};x) = U_w(x)\Phi_R(\{b_i\},\{c_j\};x) \quad \text{for } w \in W. \tag{5}$$

The Weyl denominator $\Delta_R(x)$ changes by the action of W as

$$w\Delta_R(x) = \operatorname{sgn} w\ \Delta_R(x). \tag{6}$$

For $z \in E_{\mathbf{C}}$, we now define the *Jackson integral associated with R* as

$$J_R(\{b_i\},\{c_j\};L;z) := \sum_{\chi \in L} \Phi_R(\{b_i\},\{c_j\};z+\chi)\Delta_R(z+\chi). \tag{7}$$

By definition, the Jackson integral $J_R(\{b_i\},\{c_j\};L;z)$ is obviously invariant under the shift $z \to z+\chi$ for $\chi \in L$:

$$J_R(\{b_i\},\{c_j\};L;z+\chi) = J_R(\{b_i\},\{c_j\};L;z). \tag{8}$$

For the subsequent sections, we state some facts about the Jackson integral $J_R(\{b_i\},\{c_j\};L;z)$ (see [1,7,8,10] for the detail). Let $\Theta_R(\{b_i\},\{c_j\};z)$ be a function defined by

$$\Theta_R(\{b_i\},\{c_j\};z) := \prod_{\substack{\alpha>0 \\ \alpha:\text{short}}} \frac{q^{\left(\frac{s-1}{2}-\sum_{i=1}^{s} b_i\right)\langle\alpha,z\rangle}\theta(q^{\langle\alpha,z\rangle})}{\prod_{i=1}^{s}\theta(q^{b_i+\langle\alpha,z\rangle})} \prod_{\substack{\alpha>0 \\ \alpha:\text{long}}} \frac{q^{\left(\frac{l-1}{2}-\sum_{j=1}^{l} c_j\right)\langle\alpha,z\rangle}\theta(q^{\langle\alpha,z\rangle})}{\prod_{j=1}^{l}\theta(q^{c_j+\langle\alpha,z\rangle})}.$$

The following propositions hold for $L = P$ or Q:

Proposition 1. *For $L = P$ or Q, the sum $J_R(\{b_i\},\{c_j\};L;z)$ is expressed as*

$$J_R(\{b_i\},\{c_j\};L;z) = C_R(\{b_i\},\{c_j\};L)\Theta_R(\{b_i\},\{c_j\};z), \tag{9}$$

where $C_R(\{b_i\},\{c_j\};L)$ is a constant not depending on $z \in E_{\mathbf{C}}$, if and only if
$s = 1$ for A_n, D_n, E_6, E_7 and E_8-type,
$(s,l) = (1,1)$ or $(2n-1,0)$ for B_n-type,
$(s,l) = (1,1)$ or $(0,\frac{n+1}{2})$ for C_n-type if n is odd,
$(s,l) = (1,1)$ or $(4,0)$ for G_2-type,
$(s,l) = (1,1)$ or $(3,0)$ for F_4-type.

Proposition 2. *Assume that (s,l) satisfies the condition in Proposition 1. Then the following relation holds for $L = P$ or Q:*

$$J_R(\{b_i\},\{c_j\};P;z) = |P/Q| J_R(\{b_i\},\{c_j\};Q;z).$$

In particular,

$$C_R(\{b_i\},\{c_j\};P) = |P/Q| C_R(\{b_i\},\{c_j\};Q),$$

where $|P/Q|$ is the order of the fundamental group P/Q of R, so that

R	A_n	B_n, C_n, E_7	D_n	E_6	G_2, F_4, E_8
$\lvert P/Q\rvert$	$n+1$	2	4	3	1

Proposition 3. *If $s=1$ or $(s,l)=(1,1)$, the constant $C_R(b_1,c_1;Q)$ is expressed as*

$$C_R(b_1,c_1;Q)=\prod_{\substack{\alpha>0\\ \alpha:\text{short}}}\frac{(q^{1-\langle\rho_k,\alpha^\vee\rangle-b_1})_\infty(q^{\delta_\alpha-\langle\rho_k,\alpha^\vee\rangle+b_1})_\infty}{(q^{1-\langle\rho_k,\alpha^\vee\rangle})_\infty(q^{-\langle\rho_k,\alpha^\vee\rangle})_\infty}\times\prod_{\substack{\alpha>0\\ \alpha:\text{long}}}\frac{(q^{1-\langle\rho_k,\alpha^\vee\rangle-c_1})_\infty(q^{\delta_\alpha-\langle\rho_k,\alpha^\vee\rangle+c_1})_\infty}{(q^{1-\langle\rho_k,\alpha^\vee\rangle})_\infty(q^{-\langle\rho_k,\alpha^\vee\rangle})_\infty},$$

where $2\rho_k\coloneqq b_1\sum_{\substack{\alpha>0\\ \alpha:\text{short}}}\alpha+c_1\sum_{\substack{\alpha>0\\ \alpha:\text{long}}}\alpha$ and $\delta_\alpha=1$ if $\langle\rho_k,\alpha^\vee\rangle=b_1$ or c_1, and $\delta_\alpha=0$ otherwise.

3. Convergence of Jackson integral

Let $\{\chi_1,\dots,\chi_n\}$ be the set of the fundamental coweights, i.e., $\langle\alpha_i,\chi_j\rangle=\delta_{ij}$ for all $i,j=1,\dots,n$, where δ_{ij} is the Kronecker delta.

Theorem 4. *The sum $J_R(\{b_i\},\{c_j\};L;z)$ converges if b_i and c_j satisfy*

$$\mathrm{Re}\left(\left(\frac{1-s}{2}+\sum_{i=1}^{s}b_i\right)\sum_{\substack{\alpha>0\\ \alpha:\text{short}}}\langle\alpha,\chi_k\rangle+\left(\frac{1-l}{2}+\sum_{j=1}^{l}c_j\right)\sum_{\substack{\alpha>0\\ \alpha:\text{long}}}\langle\alpha,\chi_k\rangle\right)<0$$

for $k=1,\dots,n$.

Proof. For simplicity we abbreviate $\Phi_R(\{b_i\},\{c_j\};x)$ to $\Phi_R(x)$. We denote by D the set of dominant coweights defined by

$$D\coloneqq\{\chi\in P;\langle\alpha_i,\chi\rangle\geqslant 0\text{ for }i=1,\dots,n\}.\tag{10}$$

Then we have $L\subset P=\bigcup_{w\in W}wD$. This implies that

$$|J_R(\{b_i\},\{c_j\};L;z)|<\sum_{\chi\in P}|\Phi_R(z+\chi)\Delta_R(z+\chi)|<\sum_{w\in W}\sum_{\chi\in wD}|\Phi_R(z+\chi)\Delta_R(z+\chi)|.\tag{11}$$

By using the quasi-W-symmetry (5) of $\Phi_R(x)$, it follows that

$$\begin{aligned}
&\sum_{\chi\in wD} |\Phi_R(z+\chi)\Delta_R(z+\chi)| \\
&\quad= \sum_{\chi\in D} |\Phi_R(z+w^{-1}\chi)\Delta_R(z+w^{-1}\chi)| \\
&\quad= \sum_{\chi\in D} |w\Phi_R(wz+\chi)w\Delta_R(wz+\chi)| \\
&\quad< |U_w(wz)|\sum_{\chi\in D} |\Phi_R(wz+\chi)\Delta_R(wz+\chi)|. \qquad (12)
\end{aligned}$$

From (11), (12), it is sufficient to establish that

$$\sum_{\chi\in D} |\Phi_R(z+\chi)\Delta_R(z+\chi)| \qquad (13)$$

converges. By definitions (1) and (2), we have

$$\begin{aligned}
\Phi_R(x)\Delta_R(x) = {} & q^{\left(\frac{s-1}{2}-\sum_{i=1}^{s} b_i\right)\sum_{\alpha:\mathrm{short}>0}\langle\alpha,x\rangle+\left(\frac{l-1}{2}-\sum_{j=1}^{l} c_j\right)\sum_{\alpha:\mathrm{long}>0}\langle\alpha,x\rangle} \\
& \times \prod_{i=1}^{s}\prod_{\substack{\alpha>0\\ \alpha:\mathrm{short}}} \frac{(q^{1-b_i+\langle\alpha,x\rangle})_\infty}{(q^{b_i+\langle\alpha,x\rangle})_\infty} \prod_{j=1}^{l}\prod_{\substack{\alpha>0\\ \alpha:\mathrm{long}}} \frac{(q^{1-c_j+\langle\alpha,x\rangle})_\infty}{(q^{c_j+\langle\alpha,x\rangle})_\infty} \\
& \times \prod_{\alpha>0} (q^{\langle\alpha,x\rangle}-1). \qquad (14)
\end{aligned}$$

When $\chi\in D$, from the explicit expression (14) of $\Phi_R(x)\Delta_R(x)$, it follows that the factor

$$\left|\prod_{i=1}^{s}\prod_{\substack{\alpha>0\\ \alpha:\mathrm{short}}} \frac{(q^{1-b_i+\langle\alpha,z+\chi\rangle})_\infty}{(q^{b_i+\langle\alpha,z+\chi\rangle})_\infty} \prod_{j=1}^{l}\prod_{\substack{\alpha>0\\ \alpha:\mathrm{long}}} \frac{(q^{1-c_j+\langle\alpha,z+\chi\rangle})_\infty}{(q^{c_j+\langle\alpha,z+\chi\rangle})_\infty} \prod_{\alpha>0} (q^{\langle\alpha,z+\chi\rangle}-1)\right|$$

in $|\Phi_R(z+\chi)\Delta_R(z+\chi)|$ is bounded. Hence it is sufficient to establish the convergence of the following part in (13):

$$\begin{aligned}
&\sum_{\chi\in D} \left|q^{\left(\frac{s-1}{2}-\sum_{i=1}^{s} b_i\right)\sum_{\alpha:\mathrm{short}>0}\langle\alpha,\chi\rangle+\left(\frac{l-1}{2}-\sum_{j=1}^{l} c_j\right)\sum_{\alpha:\mathrm{long}>0}\langle\alpha,\chi\rangle}\right| \\
&\quad= \sum_{\nu_1,\dots,\nu_n=0}^{\infty} \left|\prod_{k=1}^{n}\left(q^{\left(\frac{s-1}{2}-\sum_{i=1}^{s} b_i\right)\sum_{\alpha:\mathrm{short}>0}\langle\alpha,\chi_k\rangle+\left(\frac{l-1}{2}-\sum_{j=1}^{l} c_j\right)\sum_{\alpha:\mathrm{long}>0}\langle\alpha,\chi_k\rangle}\right)^{\nu_k}\right| \\
&\quad= \sum_{\nu_1,\dots,\nu_n=0}^{\infty} \prod_{k=1}^{n}\left(q^{\mathrm{Re}\left(\left(\frac{s-1}{2}-\sum_{i=1}^{s} b_i\right)\sum_{\alpha:\mathrm{short}>0}\langle\alpha,\chi_k\rangle+\left(\frac{l-1}{2}-\sum_{j=1}^{l} c_j\right)\sum_{\alpha:\mathrm{long}>0}\langle\alpha,\chi_k\rangle\right)}\right)^{\nu_k}. \qquad (15)
\end{aligned}$$

If the condition

$$\mathrm{Re}\left(\left(\frac{s-1}{2}-\sum_{i=1}^{s} b_i\right)\sum_{\substack{\alpha>0\\ \alpha:\text{short}}}\langle\alpha,\chi_k\rangle+\left(\frac{l-1}{2}-\sum_{j=1}^{l} c_j\right)\sum_{\substack{\alpha>0\\ \alpha:\text{long}}}\langle\alpha,\chi_k\rangle\right)>0$$

is satisfied, then (15) converges. This completes the proof. □

3.1. Examples

Throughout this section, let $\{\varepsilon_1,\dots,\varepsilon_n\}$ be the standard basis of $\mathbf{R}^n$ satisfying $\langle\varepsilon_i,\varepsilon_j\rangle=\delta_{ij}$ for all $i,j=1,\dots,n$.

3.1.1. B_n-type

- $$\begin{cases}\text{Basis}: \alpha_1=\varepsilon_1-\varepsilon_2,\ \alpha_2=\varepsilon_2-\varepsilon_3,\dots,\ \alpha_{n-1}=\varepsilon_{n-1}-\varepsilon_n,\ \alpha_n=\varepsilon_n,\\ \text{Fundamental coweights}: \chi_1=\varepsilon_1,\ \chi_2=\varepsilon_1+\varepsilon_2,\ \chi_3=\varepsilon_1+\varepsilon_2+\varepsilon_3,\\ \quad\dots,\ \chi_n=\varepsilon_1+\varepsilon_2+\varepsilon_3+\cdots+\varepsilon_n.\end{cases}$$

- $$\begin{cases}\text{Positive short roots}: \varepsilon_i\ (1\leqslant i\leqslant n),\\ \text{Positive long roots}: \varepsilon_i\pm\varepsilon_j\ (1\leqslant i<j\leqslant n).\end{cases}$$

- $$\begin{cases}\text{Highest root}: \varepsilon_1+\varepsilon_2=\alpha_1+2\alpha_2+2\alpha_3+\cdots+2\alpha_n\quad(\text{for Lemmas 9, 10}),\\ \text{Highest root in short roots}: \varepsilon_1=\alpha_1+\alpha_2+\cdots+\alpha_4.\end{cases}$$

The sums of the positive short roots and the positive long roots are the following:

$$\sum_{\substack{\alpha>0\\ \alpha:\text{short}}}\alpha=\sum_{i=1}^{n}\varepsilon_i,\quad \sum_{\substack{\alpha>0\\ \alpha:\text{long}}}\alpha=2\sum_{i=1}^{n}(n-i)\varepsilon_i,$$

so that we have

$$\sum_{\substack{\alpha>0\\ \alpha:\text{short}}}\langle\alpha,\chi_k\rangle=k,\quad \sum_{\substack{\alpha>0\\ \alpha:\text{long}}}\langle\alpha,\chi_k\rangle=k(2n-1-k).$$

By Theorem 4, if $(s,l)=(1,1)$, a sufficient condition for convergence is

$$\mathrm{Re}(b_1+(n-1)c_1)<0\quad\text{and}\quad\mathrm{Re}(b_1+2(n-1)c_1)<0,$$

as we see from [7]. And if $(s,l)=(2n-1,0)$, we have

$$\mathrm{Re}(b_1+\cdots+b_{2n-1})<0,$$

which was mentioned in [6].

3.1.2. G_2-type

- $$\begin{cases} \text{Basis}: \alpha_1 = \varepsilon_1 - \varepsilon_2,\ \alpha_2 = -2\varepsilon_1 + \varepsilon_2 + \varepsilon_3, \\ \text{Fundamental coweights}: \chi_1 = 2\alpha_1 + \alpha_2,\ \chi_2 = (3\alpha_1 + 2\alpha_2)/3. \end{cases}$$

- $$\begin{cases} \text{Positive short roots}: \alpha_1,\ \alpha_1 + \alpha_2,\ 2\alpha_1 + \alpha_2, \\ \text{Positive long roots}: \alpha_2,\ 3\alpha_1 + \alpha_2,\ 3\alpha_1 + 2\alpha_2. \end{cases}$$

- $$\begin{cases} \text{Highest root}: 3\alpha_1 + 2\alpha_2 \quad (\text{for Lemmas 9, 10}), \\ \text{Highest root in short roots}: 2\alpha_1 + \alpha_2. \end{cases}$$

The sums of the positive short roots and the positive long roots are the following:

$$\sum_{\substack{\alpha>0 \\ \alpha:\text{short}}} \alpha = 4\alpha_1 + 2\alpha_2 \quad \sum_{\substack{\alpha>0 \\ \alpha:\text{long}}} \alpha = 6\alpha_1 + 4\alpha_2,$$

so that we have

$$\sum_{\substack{\alpha>0 \\ \alpha:\text{short}}} \langle \alpha, \chi_1 \rangle = 4, \quad \sum_{\substack{\alpha>0 \\ \alpha:\text{short}}} \langle \alpha, \chi_2 \rangle = 2, \quad \sum_{\substack{\alpha>0 \\ \alpha:\text{long}}} \langle \alpha, \chi_1 \rangle = 6, \quad \sum_{\substack{\alpha>0 \\ \alpha:\text{long}}} \langle \alpha, \chi_2 \rangle = 4.$$

By Theorem 4, if $(s,l) = (1,1)$, we have a convergence condition as

$$\mathrm{Re}(2b_1 + 3c_1) < 0 \quad \text{and} \quad \mathrm{Re}(b_1 + 2c_1) < 0,$$

as we see in [7]. And if $(s,l) = (4,0)$, we have

$$\mathrm{Re}(2(b_1 + b_2 + b_3 + b_4) - 1) < 0, \tag{16}$$

which was mentioned in [5].

3.1.3. F_4-type

Since the root systems F_4 and $F_4^\vee$ are isomorphic with orthogonal transformation [3, p. 806], we take a basis of $F_4^\vee$ instead of that of F_4.

- $$\begin{cases} \text{Basis}: \alpha_1 = \varepsilon_2 - \varepsilon_3,\ \alpha_2 = \varepsilon_3 - \varepsilon_4,\ \alpha_3 = 2\varepsilon_4,\ \alpha_4 = \varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4, \\ \text{Fundamental coweights}: \chi_1 = \varepsilon_1 + \varepsilon_2,\ \chi_2 = 2\varepsilon_1 + \varepsilon_2 + \varepsilon_3, \\ \quad \chi_3 = (3\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4)/2,\ \chi_4 = \varepsilon_1. \end{cases}$$

- $$\begin{cases} \text{Positive short roots}: \varepsilon_i \pm \varepsilon_j \ (1 \leqslant i < j \leqslant 4), \\ \text{Positive long roots}: 2\varepsilon_i \ (1 \leqslant i \leqslant 4), \quad \varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4. \end{cases}$$

- $$\begin{cases} \text{Highest root}: 2\varepsilon_1 = 2\alpha_1 + 4\alpha_2 + 3\alpha_3 + 2\alpha_4 \quad (\text{for Lemmas 9, 10}), \\ \text{Highest root in short roots}: \varepsilon_1 + \varepsilon_2 = 2\alpha_1 + 3\alpha_2 + 2\alpha_3 + \alpha_4. \end{cases}$$

The sums of the positive short roots and the positive long roots are the following:

$$\sum_{\substack{\alpha>0\\ \alpha:\text{short}}} \alpha = 6\varepsilon_1 + 4\varepsilon_2 + 2\varepsilon_3, \qquad \sum_{\substack{\alpha>0\\ \alpha:\text{long}}} \alpha = 10\varepsilon_1 + 2\varepsilon_2 + 2\varepsilon_3 + 2\varepsilon_4,$$

so that, we have

$$\sum_{\substack{\alpha>0\\ \alpha:\text{short}}} \langle \alpha, \chi_1 \rangle = 10, \quad \sum_{\substack{\alpha>0\\ \alpha:\text{short}}} \langle \alpha, \chi_2 \rangle = 18, \quad \sum_{\substack{\alpha>0\\ \alpha:\text{short}}} \langle \alpha, \chi_3 \rangle = 12,$$

$$\sum_{\substack{\alpha>0\\ \alpha:\text{short}}} \langle \alpha, \chi_4 \rangle = 6, \quad \sum_{\substack{\alpha>0\\ \alpha:\text{long}}} \langle \alpha, \chi_1 \rangle = 12, \quad \sum_{\substack{\alpha>0\\ \alpha:\text{long}}} \langle \alpha, \chi_2 \rangle = 24,$$

$$\sum_{\substack{\alpha>0\\ \alpha:\text{long}}} \langle \alpha, \chi_3 \rangle = 18, \quad \sum_{\substack{\alpha>0\\ \alpha:\text{long}}} \langle \alpha, \chi_4 \rangle = 10.$$

From Theorem 4, if $(s,l) = (1,1)$, we have a convergence condition as

$$\mathrm{Re}(5b_1 + 6c_1) < 0 \quad \text{and} \quad \mathrm{Re}(3b_1 + 5c_1) < 0.$$

And if $(s,l) = (3,0)$, we have

$$\mathrm{Re}(6(b_1 + b_2 + b_3) - 1) < 0, \text{which will be used in Theorem A.3.} \tag{17}$$

4. Asymptotic behavior

Following [10], define a sum $M_R(\{b_i\},\{c_j\};L;z)$ over a lattice L as

$$M_R(\{b_i\},\{c_j\};L;z) \coloneqq \sum_{\chi\in L} \Psi_R(\{b_i\},\{c_j\};z+\chi), \tag{18}$$

where

$$\Psi_R(\{b_i\},\{c_j\};x) \coloneqq \prod_{\substack{\alpha\in R\\ \alpha:\text{short}}} \frac{\prod_{i=1}^{s} (q^{1-b_i+\langle \alpha,x\rangle})_\infty}{(q^{1+\langle \alpha,x\rangle})_\infty} \prod_{\substack{\alpha\in R\\ \alpha:\text{long}}} \frac{\prod_{j=1}^{l} (q^{1-c_j+\langle \alpha,x\rangle})_\infty}{(q^{1+\langle \alpha,x\rangle})_\infty}.$$

We call $M_R(\{b_i\},\{c_j\};L;z)$ the *Macdonald type sum*. If (s,l) is in the list of Proposition 1, for $L = P$ or Q, we easily see that the sum $M_R(\{b_i\},\{c_j\};L;z)$ does not depend on $z\in E_{\mathbf{C}}$ and coincides with the constant $C_R(\{b_i\},\{c_j\};L)$:

$$M_R(\{b_i\},\{c_j\};L;z) = \frac{J_R(\{b_i\},\{c_j\};L;z)}{\Theta_R(\{b_i\},\{c_j\};z)} = C_R(\{b_i\},\{c_j\};L). \tag{19}$$

From (19) and Proposition 2, obviously we have

$$M_R(\{b_i\},\{c_j\};P;z) = |P/Q| M_R(\{b_i\},\{c_j\};Q;z). \tag{20}$$

We also define a sum $M_R(L;z)$ over a lattice L as

$$M_R(L;z) := \sum_{\chi\in L}\left(\prod_{\alpha\in R}\frac{1}{(q^{1+\langle\alpha,z+\chi\rangle})_\infty}\right),$$

which does not depend on $z\in E_{\mathbf{C}}$ if $L=P$ or Q.

Proposition 5. *The following relations hold for $L=P$ or Q*:

$$M_R(P;z)=|P/Q|M_R(Q;z) \quad \text{and} \quad M_R(Q;z)=(q)_\infty^n.$$

Proof. See (57). $\square$

We assume (s,l) satisfies the condition in Proposition 1 under here. By Theorem 4, the Macdonald type sum $M_R(\{b_i\},\{c_j\};L;z)$ still has its meaning when b_i and c_j are sufficiently negative.

Theorem 6. *The Macdonald type sum $M_R(\{b_i-N\},\{c_j-N\};L;z)$ at $N\to+\infty$ is the following*:

$$\lim_{N\to+\infty} M_R(\{b_i-N\},\{c_j-N\};Q;z)=(q)_\infty^n,$$
$$\lim_{N\to+\infty} M_R(\{b_i-N\},\{c_j-N\};P;z)=|P/Q|(q)_\infty^n.$$

The following follows from Theorem 6 immediately:

Corollary 7. *The asymptotic behavior of the Jackson integral $J_R(\{b_i-N\},\{c_j-N\};L;z)$ at $N\to+\infty$ is following*:

$$\begin{aligned}&J_R(\{b_i-N\},\{c_j-N\};Q;z)\\ &\quad\sim(-1)^{(sR_1+lR_2)N}q^{(sR_1+lR_2)N(N+1)/2-(b_1+\cdots+b_s)R_1N-(c_1+\cdots+c_s)R_2N}\\ &\qquad\times(q)_\infty^n\Theta_R(\{b_i\},\{c_j\};z),\end{aligned}$$

$$\begin{aligned}&J_R(\{b_i-N\},\{c_j-N\};P;z)\\ &\quad\sim(-1)^{(sR_1+lR_2)N}q^{(sR_1+lR_2)N(N+1)/2-(b_1+\cdots+b_s)R_1N-(c_1+\cdots+c_s)R_2N}\\ &\qquad\times|P/Q|(q)_\infty^n\Theta_R(\{b_i\},\{c_j\};z),\end{aligned}$$

where R_1 and R_2 are the number of all short and long positive roots respectively.

Before proving Theorem 6, we establish four lemmas.

Lemma 8. *Let D_N be the set defined by*

$$D_N := \begin{cases} \{\chi \in P; |\langle \alpha, \chi \rangle| \leqslant N \text{ for all } \alpha \in R\} & \text{if } l \neq 0, \\ \{\chi \in P; |\langle \alpha, \chi \rangle| \leqslant N \text{ for all short } \alpha \in R\} & \text{if } l = 0. \end{cases} \tag{21}$$

Then,

$$\lim_{N \to +\infty} \sum_{\chi \in D_N} \Psi_R(\{b_i - N\}, \{c_j - N\}; z + \chi) = M_R(P; z).$$

Proof. We denote by $F(\chi; N)$ and $G(\chi)$ the numerator and denominator of $\Psi_R(\{b_i - N\}, \{c_j - N\}; z + \chi)$, respectively, i.e.,

$$\Psi_R(\{b_i - N\}, \{c_j - N\}; z + \chi) = \frac{F(\chi; N)}{G(\chi)},$$

where

$$F(\chi; N) := \prod_{\substack{\alpha \in R \\ \alpha:\text{short}}} \prod_{i=1}^{s} (q^{1-b_i+N+\langle \alpha, z+\chi \rangle})_\infty \prod_{\substack{\alpha \in R \\ \alpha:\text{long}}} \prod_{j=1}^{l} (q^{1-c_j+N+\langle \alpha, z+\chi \rangle})_\infty,$$

$$G(\chi) := \prod_{\alpha \in R} (q^{1+\langle \alpha, z+\chi \rangle})_\infty. \tag{22}$$

We assume ε is an arbitrary positive number. If $\chi \in D_N$, the factor $F(\chi; N)$ is bounded, so that

$$|F(\chi; N)| < C_1, \tag{23}$$

where C_1 is a constant not depending on χ and N. The factor $G(\chi)$ is written as

$$G(\chi) = \prod_{\alpha > 0} (q^{1+\langle \alpha, z+\chi \rangle})_\infty (q^{1-\langle \alpha, z+\chi \rangle})_\infty = \prod_{\alpha > 0} \theta(q^{\langle \alpha, z+\chi \rangle})/(1 - q^{\langle \alpha, z+\chi \rangle}). \tag{24}$$

Using the quasi-periodicity (4) of the function $\theta(\xi)$, we have

$$\frac{\theta(q^{\langle \alpha, z \rangle})}{\theta(q^{\langle \alpha, z+\chi \rangle})} = (-1)^{\langle \alpha, \chi \rangle} q^{\frac{1}{2}\langle \alpha, \chi \rangle^2 + (\langle \alpha, z \rangle - \frac{1}{2})\langle \alpha, \chi \rangle}. \tag{25}$$

From (24) and (25), it follows that

$$G(\chi)^{-1} = \prod_{\alpha > 0} (-1)^{\langle \alpha, \chi \rangle} q^{\frac{1}{2}\langle \alpha, \chi \rangle^2 + (\langle \alpha, z \rangle - \frac{1}{2})\langle \alpha, \chi \rangle} (1 - q^{\langle \alpha, z+\chi \rangle})/\theta(q^{\langle \alpha, z \rangle}). \tag{26}$$

By using the Weyl denominator formula for (26), we have

$$\begin{aligned} |G(\chi)^{-1}| &= \left| \prod_{\alpha > 0} (q^{\frac{1}{2}\langle \alpha, z+\chi \rangle} - q^{-\frac{1}{2}\langle \alpha, z+\chi \rangle}) q^{\frac{1}{2}\langle \alpha, \chi \rangle^2 + \langle \alpha, z \rangle \langle \alpha, \chi \rangle} q^{\frac{1}{2}\langle \alpha, z \rangle} / \theta(q^{\langle \alpha, z \rangle}) \right| \\ &< C_2 \left| \left(\sum_{w \in W} \operatorname{sgn} w q^{\langle w\rho, z+\chi \rangle} \right) q^{\sum_{\alpha > 0} \frac{1}{2}\langle \alpha, \chi \rangle^2 + \sum_{\alpha > 0} \langle \alpha, z \rangle \langle \alpha, \chi \rangle} \right| \\ &< C_2 \sum_{w \in W} |q^{\sum_{\alpha > 0} \frac{1}{2}\langle \alpha, \chi \rangle^2 + \sum_{\alpha > 0} \langle \alpha, z \rangle \langle \alpha, \chi \rangle + \langle w\rho, z+\chi \rangle}|, \end{aligned} \tag{27}$$

where $\rho = \frac{1}{2}\sum_{\alpha>0}\alpha$ and $C_2 := |\prod_{\alpha>0} q^{\frac{1}{2}\langle\alpha,z\rangle}/\theta(q^{\langle\alpha,z\rangle})|$. Since the quadratic part

$$\frac{1}{2}\sum_{\alpha>0}\langle\alpha,\chi\rangle^2$$

in (27) is positive definite for $\chi\in P$, there exists a positive integer m_1 such that

$$\sum_{w\in W}|q^{\sum_{\alpha>0}\frac{1}{2}\langle\alpha,\chi\rangle^2+\sum_{\alpha>0}\langle\alpha,z\rangle\langle\alpha,\chi\rangle+\langle w\rho,z+\chi\rangle}| \quad < q^{\frac{1}{4}\sum_{\alpha>0}\langle\alpha,\chi\rangle^2} \tag{28}$$

for $\chi\notin\{\chi\in P; |\langle\alpha_i,\chi\rangle|<m_1 \text{ for all } i=1,\ldots,n\}$. Since the sum

$$\sum_{\chi\in P} q^{\frac{1}{4}\sum_{\alpha>0}\langle\alpha,\chi\rangle^2}$$

converges, there exists a positive integer m_2 such that

$$\sum_{\chi\in P-M_2} q^{\frac{1}{4}\sum_{\alpha>0}\langle\alpha,\chi\rangle^2}<\frac{\varepsilon}{3C_1C_2}, \tag{29}$$

where $M_2=\{\chi\in P; |\langle\alpha_i,\chi\rangle|<m_2 \text{ for all } i=1,\ldots,n\}$. We set

$$M:=\{\chi\in P; |\langle\alpha_i,\chi\rangle|<\max\{m_1,m_2\} \text{ for all } i=1,\ldots,n\},$$

which does not depend on N. Then, by (27)–(29), we have

$$\sum_{\chi\in P-M}|G(\chi)^{-1}|<\frac{\varepsilon}{3C_1}. \tag{30}$$

From (23) and (30), it follows that

$$\sum_{\chi\in D_N-M}|F(\chi;N)G(\chi)^{-1}|<C_1\sum_{\chi\in D_N-M}|G(\chi)^{-1}|<\varepsilon/3. \tag{31}$$

For $\chi\in M$, there exists N_0 such that

$$|F(\chi;N)G(\chi)^{-1}-G(\chi)^{-1}|<\frac{\varepsilon}{3|M|} \tag{32}$$

for all $N>N_0$. Hence, from (30)–(32), we obtain

$$\begin{aligned}&\left|\sum_{\chi\in D_N}F(\chi;N)G(\chi)^{-1}-\sum_{\chi\in P}G(\chi)^{-1}\right|\\ &\quad<\sum_{\chi\in M}|F(\chi;N)G(\chi)^{-1}-G(\chi)^{-1}|+\sum_{\chi\in D_N-M}|F(\chi;N)G(\chi)^{-1}|\\ &\qquad+\sum_{\chi\in P-M}|G(\chi)^{-1}|<\varepsilon. \quad \square\end{aligned}$$

We still use notation in the proof of Lemma 8. Let D be the set of dominant coweights defined by (10).

Lemma 9. *Let $\tilde{\alpha}$ be the positive root defined by*

$$\tilde{\alpha} := \begin{cases} \text{the highest root} & \text{if } l \neq 0, \\ \text{the highest root in short roots} & \text{if } l = 0. \end{cases}$$

(*See Examples in Section* 3.) *For a sufficiently large positive integer N and $\chi \in D - D_N$, there exists a constant $C > 0$ not depending on N and χ such that*

$$|\Psi_R(\{b_i - N\}, \{c_j - N\}; z + \chi)| < C q^{\frac{N}{4}\langle \tilde{\alpha}, \chi \rangle}.$$

Proof. For $\chi \in D - D_N$, we divide R^+ into two sets as follows:

$$R^+ = A_\chi^+ \cup B_\chi^+, \tag{33}$$

where $A_\chi^+ := \{\alpha \in R^+; 0 \leqslant \langle \alpha, \chi \rangle \leqslant N\}$ and $B_\chi^+ := \{\alpha \in R^+; N < \langle \alpha, \chi \rangle\}$. By definitions (10) and (21), $D - D_N$ is described as

$$D - D_N = \{\chi \in D; N < \langle \tilde{\alpha}, \chi \rangle\}. \tag{34}$$

From (34), it is obvious that

$$\tilde{\alpha} \in B_\chi^+ \quad \text{if } \chi \in D - D_N. \tag{35}$$

From (22) and (33), it follows that

$$\begin{aligned} &F(\chi; N) \\ &= \prod_{\substack{\alpha > 0 \\ \alpha:\text{short}}} \prod_{i=1}^{s} (q^{1-b_i+N+\langle \alpha, z+\chi \rangle})_\infty \prod_{\substack{\alpha > 0 \\ \alpha:\text{long}}} \prod_{j=1}^{l} (q^{1-c_j+N+\langle \alpha, z+\chi \rangle})_\infty \\ &\quad \times \prod_{\substack{\alpha \in A_\chi^+ \\ \alpha:\text{short}}} \prod_{i=1}^{s} (q^{1-b_i+N-\langle \alpha, z+\chi \rangle})_\infty \prod_{\substack{\alpha \in A_\chi^+ \\ \alpha:\text{long}}} \prod_{j=1}^{l} (q^{1-c_j+N-\langle \alpha, z+\chi \rangle})_\infty \\ &\quad \times \prod_{\substack{\alpha \in B_\chi^+ \\ \alpha:\text{short}}} \prod_{i=1}^{s} (q^{1-b_i+N-\langle \alpha, z+\chi \rangle})_\infty \prod_{\substack{\alpha \in B_\chi^+ \\ \alpha:\text{long}}} \prod_{j=1}^{l} (q^{1-c_j+N-\langle \alpha, z+\chi \rangle})_\infty. \end{aligned} \tag{36}$$

The last factor appearing in (36) is equal to the following:

$$\begin{aligned} &\prod_{\substack{\alpha \in B_\chi^+ \\ \alpha:\text{short}}} \prod_{i=1}^{s} (q^{1-b_i+N-\langle \alpha, z+\chi \rangle})_\infty \prod_{\substack{\alpha \in B_\chi^+ \\ \alpha:\text{long}}} \prod_{j=1}^{l} (q^{1-c_j+N-\langle \alpha, z+\chi \rangle})_\infty \\ &= \prod_{\substack{\alpha \in B_\chi^+ \\ \alpha:\text{short}}} \prod_{i=1}^{s} \frac{\theta(q^{b_i+\langle \alpha, z \rangle+\langle \alpha, \chi \rangle-N})}{(q^{b_i+\langle \alpha, z \rangle+\langle \alpha, \chi \rangle-N})_\infty} \prod_{\substack{\alpha \in B_\chi^+ \\ \alpha:\text{long}}} \prod_{j=1}^{l} \frac{\theta(q^{c_j+\langle \alpha, z \rangle+\langle \alpha, \chi \rangle-N})}{(q^{c_j+\langle \alpha, z \rangle+\langle \alpha, \chi \rangle-N})_\infty} \end{aligned}$$

$$= \prod_{\substack{\alpha\in B_\chi^+\\ \alpha:\text{short}}} \prod_{i=1}^{s} (-1)^{\langle\alpha,\chi\rangle} q^{-\frac{1}{2}(\langle\alpha,\chi\rangle-N)^2-(b_i+\langle\alpha,z\rangle-\frac{1}{2})(\langle\alpha,\chi\rangle-N)}$$

$$\times \prod_{\substack{\alpha\in B_\chi^+\\ \alpha:\text{long}}} \prod_{j=1}^{l} (-1)^{\langle\alpha,\chi\rangle} q^{-\frac{1}{2}(\langle\alpha,\chi\rangle-N)^2-(c_j+\langle\alpha,z\rangle-\frac{1}{2})(\langle\alpha,\chi\rangle-N)}$$

$$\times \prod_{\substack{\alpha\in B_\chi^+\\ \alpha:\text{short}}} \prod_{i=1}^{s} \frac{\theta(q^{b_i+\langle\alpha,z\rangle})}{(q^{b_i+\langle\alpha,z\rangle+\langle\alpha,\chi\rangle-N})_\infty} \prod_{\substack{\alpha\in B_\chi^+\\ \alpha:\text{long}}} \prod_{j=1}^{l} \frac{\theta(q^{c_j+\langle\alpha,z\rangle})}{(q^{c_j+\langle\alpha,z\rangle+\langle\alpha,\chi\rangle-N})_\infty}. \tag{37}$$

From (36) and (37), since the factors

$$\prod_{\substack{\alpha>0\\ \alpha:\text{short}}} \prod_{i=1}^{s} (q^{1-b_i+N+\langle\alpha,z+\chi\rangle})_\infty \prod_{\substack{\alpha>0\\ \alpha:\text{long}}} \prod_{j=1}^{l} (q^{1-c_j+N+\langle\alpha,z+\chi\rangle})_\infty,$$

$$\prod_{\substack{\alpha\in A_\chi^+\\ \alpha:\text{short}}} \prod_{i=1}^{s} (q^{1-b_i+N-\langle\alpha,z+\chi\rangle})_\infty \prod_{\substack{\alpha\in A_\chi^+\\ \alpha:\text{long}}} \prod_{j=1}^{l} (q^{1-c_j+N-\langle\alpha,z+\chi\rangle})_\infty$$

and

$$\prod_{\substack{\alpha\in B_\chi^+\\ \alpha:\text{short}}} \prod_{i=1}^{s} \frac{\theta(q^{b_i+\langle\alpha,z\rangle})}{(q^{b_i+\langle\alpha,z\rangle+\langle\alpha,\chi\rangle-N})_\infty} \prod_{\substack{\alpha\in B_\chi^+\\ \alpha:\text{long}}} \prod_{j=1}^{l} \frac{\theta(q^{c_j+\langle\alpha,z\rangle})}{(q^{c_j+\langle\alpha,z\rangle+\langle\alpha,\chi\rangle-N})_\infty}$$

appearing in $F(\chi;N)$ are bounded for $\chi\in D-D_N$, there exists a constant C_3 not depending on χ and N such that

$$|F(\chi;N)| < C_3 \left| \prod_{\substack{\alpha\in B_\chi^+\\ \alpha:\text{short}}} \prod_{i=1}^{s} q^{-\frac{1}{2}(\langle\alpha,\chi\rangle-N)^2-(b_i+\langle\alpha,z\rangle-\frac{1}{2})(\langle\alpha,\chi\rangle-N)} \right.$$

$$\left. \times \prod_{\substack{\alpha\in B_\chi^+\\ \alpha:\text{long}}} \prod_{j=1}^{l} q^{-\frac{1}{2}(\langle\alpha,\chi\rangle-N)^2-(c_j+\langle\alpha,z\rangle-\frac{1}{2})(\langle\alpha,\chi\rangle-N)} \right|. \tag{38}$$

For $G(\chi)^{-1}$, the factor $(1-q^{\langle\alpha,z+\chi\rangle})/\theta(q^{\langle\alpha,z\rangle})$ in (26) is bounded if $\chi\in D$. Then there exists a constant C_4 not depending on χ such that

$$|G(\chi)^{-1}| < C_4 \left| \prod_{\alpha>0} q^{\frac{1}{2}\langle\alpha,\chi\rangle^2+(\langle\alpha,z\rangle-\frac{1}{2})\langle\alpha,\chi\rangle} \right|. \tag{39}$$

Since (s,l) satisfies the condition in Proposition 1, for $i,j=1,\dots,n$, we have

$$(s-1)\sum_{\substack{\alpha>0\\ \alpha:\text{short}}}\langle\alpha,\chi_i\rangle\langle\alpha,\chi_j\rangle+(l-1)\sum_{\substack{\alpha>0\\ \alpha:\text{long}}}\langle\alpha,\chi_i\rangle\langle\alpha,\chi_j\rangle=0. \tag{40}$$

(See [8, p. 336], (s,l) in Proposition 1 was chosen to satisfy (40).) This implies that

$$\prod_{\alpha>0}q^{\frac{1}{2}\langle\alpha,\chi\rangle^2}=\prod_{\substack{\alpha>0\\ \alpha:\text{short}}}q^{\frac{s}{2}\langle\alpha,\chi\rangle^2}\prod_{\substack{\alpha>0\\ \alpha:\text{long}}}q^{\frac{l}{2}\langle\alpha,\chi\rangle^2} \tag{41}$$

and

$$\prod_{\alpha>0}q^{\langle\alpha,z\rangle\langle\alpha,\chi\rangle}=\prod_{\substack{\alpha>0\\ \alpha:\text{short}}}q^{s\langle\alpha,z\rangle\langle\alpha,\chi\rangle}\prod_{\substack{\alpha>0\\ \alpha:\text{long}}}q^{l\langle\alpha,z\rangle\langle\alpha,\chi\rangle}. \tag{42}$$

By virtue of the convergence condition in Theorem 4, we obtain

$$\left|\prod_{\alpha>0}q^{-\frac{1}{2}\langle\alpha,\chi\rangle}\right|<\left|\prod_{\substack{\alpha>0\\ \alpha:\text{short}}}q^{(b_1+\cdots+b_s-\frac{s}{2})\langle\alpha,\chi\rangle}\prod_{\substack{\alpha>0\\ \alpha:\text{long}}}q^{(c_1+\cdots+c_l-\frac{l}{2})\langle\alpha,\chi\rangle}\right|. \tag{43}$$

From (39), (41)–(43), it follows that

$$\begin{aligned}
&|G(\chi)^{-1}|/C_4\\
&<\left|\prod_{\substack{\alpha>0\\ \alpha:\text{short}}}q^{\frac{s}{2}\langle\alpha,\chi\rangle^2+(b_1+\cdots+b_s-\frac{s}{2}+s\langle\alpha,z\rangle)\langle\alpha,\chi\rangle}\prod_{\substack{\alpha>0\\ \alpha:\text{long}}}q^{\frac{l}{2}\langle\alpha,\chi\rangle^2+(c_1+\cdots+c_l-\frac{l}{2}+l\langle\alpha,z\rangle)\langle\alpha,\chi\rangle}\right|\\
&=\left|\prod_{\substack{\alpha\in A_\chi^+\\ \alpha:\text{short}}}q^{\frac{s}{2}\langle\alpha,\chi\rangle^2+(b_1+\cdots+b_s-\frac{s}{2}+s\langle\alpha,z\rangle)\langle\alpha,\chi\rangle}\prod_{\substack{\alpha\in A_\chi^+\\ \alpha:\text{long}}}q^{\frac{l}{2}\langle\alpha,\chi\rangle^2+(c_1+\cdots+c_l-\frac{l}{2}+l\langle\alpha,z\rangle)\langle\alpha,\chi\rangle}\right.\\
&\quad\times\prod_{\substack{\alpha\in B_\chi^+\\ \alpha:\text{short}}}q^{\frac{s}{2}\langle\alpha,\chi\rangle^2+(b_1+\cdots+b_s-\frac{s}{2}+s\langle\alpha,z\rangle)\langle\alpha,\chi\rangle}\\
&\quad\left.\times\prod_{\substack{\alpha\in B_\chi^+\\ \alpha:\text{long}}}q^{\frac{l}{2}\langle\alpha,\chi\rangle^2+(c_1+\cdots+c_l-\frac{l}{2}+l\langle\alpha,z\rangle)\langle\alpha,\chi\rangle}\right|.
\end{aligned} \tag{44}$$

If $s \neq 0$ and $l = 0$, then it follows that

$$\begin{aligned}
&\left| \prod_{\substack{\alpha\in A_\chi^+\\ \alpha:\text{short}}} q^{\frac{s}{2}\langle\alpha,\chi\rangle^2+(b_1+\cdots+b_s-\frac{s}{2}+s\langle\alpha,z\rangle)\langle\alpha,\chi\rangle} \prod_{\substack{\alpha\in A_\chi^+\\ \alpha:\text{long}}} q^{\frac{l}{2}\langle\alpha,\chi\rangle^2+(c_1+\cdots+c_l-\frac{l}{2}+l\langle\alpha,z\rangle)\langle\alpha,\chi\rangle} \right|\\
&\quad = \prod_{\substack{\alpha\in A_\chi^+\\ \alpha:\text{short}}} q^{\frac{s}{2}\langle\alpha,\chi\rangle^2+\mathrm{Re}(b_1+\cdots+b_s-\frac{s}{2}+s\langle\alpha,z\rangle)\langle\alpha,\chi\rangle}\\
&\quad = \prod_{\substack{\alpha\in A_\chi^+\\ \alpha:\text{short}}} q^{\frac{s}{2}(\langle\alpha,\chi\rangle+\frac{1}{s}\mathrm{Re}(b_1+\cdots+b_s-\frac{s}{2}+s\langle\alpha,z\rangle))^2-\frac{1}{2s}(\mathrm{Re}(b_1+\cdots+b_s-\frac{s}{2}+s\langle\alpha,z\rangle))^2}\\
&\quad \leqslant \prod_{\substack{\alpha\in A_\chi^+\\ \alpha:\text{short}}} q^{-\frac{1}{2s}(\mathrm{Re}(b_1+\cdots+b_s-\frac{s}{2}+s\langle\alpha,z\rangle))^2}\\
&\quad < \prod_{\substack{\alpha>0\\ \alpha:\text{short}}} q^{-\frac{1}{2s}(\mathrm{Re}(b_1+\cdots+b_s-\frac{s}{2}+s\langle\alpha,z\rangle))^2}.
\end{aligned} \tag{45}$$

In the same way as (45), if $s = 0$ and $l \neq 0$, then

$$\begin{aligned}
&\left| \prod_{\substack{\alpha\in A_\chi^+\\ \alpha:\text{short}}} q^{\frac{s}{2}\langle\alpha,\chi\rangle^2+(b_1+\cdots+b_s-\frac{s}{2}+s\langle\alpha,z\rangle)\langle\alpha,\chi\rangle} \prod_{\substack{\alpha\in A_\chi^+\\ \alpha:\text{long}}} q^{\frac{l}{2}\langle\alpha,\chi\rangle^2+(c_1+\cdots+c_l-\frac{l}{2}+l\langle\alpha,z\rangle)\langle\alpha,\chi\rangle} \right|\\
&\quad < \prod_{\substack{\alpha>0\\ \alpha:\text{long}}} q^{-\frac{1}{2l}(\mathrm{Re}(c_1+\cdots+c_l-\frac{l}{2}+l\langle\alpha,z\rangle))^2}.
\end{aligned} \tag{46}$$

Moreover, if $s \neq 0$ and $l \neq 0$, then

$$\begin{aligned}
&\left| \prod_{\substack{\alpha\in A_\chi^+\\ \alpha:\text{short}}} q^{\frac{s}{2}\langle\alpha,\chi\rangle^2+(b_1+\cdots+b_s-\frac{s}{2}+s\langle\alpha,z\rangle)\langle\alpha,\chi\rangle} \prod_{\substack{\alpha\in A_\chi^+\\ \alpha:\text{long}}} q^{\frac{l}{2}\langle\alpha,\chi\rangle^2+(c_1+\cdots+c_l-\frac{l}{2}+l\langle\alpha,z\rangle)\langle\alpha,\chi\rangle} \right|\\
&\quad < \prod_{\substack{\alpha>0\\ \alpha:\text{short}}} q^{-\frac{1}{2s}(\mathrm{Re}(b_1+\cdots+b_s-\frac{s}{2}+s\langle\alpha,z\rangle))^2} \prod_{\substack{\alpha>0\\ \alpha:\text{long}}} q^{-\frac{1}{2l}(\mathrm{Re}(c_1+\cdots+c_l-\frac{l}{2}+l\langle\alpha,z\rangle))^2}.
\end{aligned} \tag{47}$$

From (44)–(47), we have

$$|G(\chi)^{-1}| < C_5 \left| \prod_{\substack{\alpha\in B_\chi^+\\ \alpha:\text{short}}} q^{\frac{s}{2}\langle\alpha,\chi\rangle^2+(b_1+\cdots+b_s-\frac{s}{2}+s\langle\alpha,z\rangle)\langle\alpha,\chi\rangle} \times \prod_{\substack{\alpha\in B_\chi^+\\ \alpha:\text{long}}} q^{\frac{l}{2}\langle\alpha,\chi\rangle^2+(c_1+\cdots+c_l-\frac{l}{2}+l\langle\alpha,z\rangle)\langle\alpha,\chi\rangle} \right|, \tag{48}$$

where C_5 is a constant not depending on χ and N such that

$$C_5/C_4 := \begin{cases} \prod_{\substack{\alpha>0\\ \alpha:\text{short}}} q^{-\frac{1}{2s}(\mathrm{Re}(b_1+\cdots+b_s-\frac{s}{2}+s\langle\alpha,z\rangle))^2} & \text{if } s\neq 0,\ l=0,\\ \prod_{\substack{\alpha>0\\ \alpha:\text{long}}} q^{-\frac{1}{2l}(\mathrm{Re}(c_1+\cdots+c_l-\frac{l}{2}+l\langle\alpha,z\rangle))^2} & \text{if } s=0,\ l\neq 0,\\ \prod_{\substack{\alpha>0\\ \alpha:\text{short}}} q^{-\frac{1}{2s}(\mathrm{Re}(b_1+\cdots+b_s-\frac{s}{2}+s\langle\alpha,z\rangle))^2} & \\ \quad\times\prod_{\substack{\alpha>0\\ \alpha:\text{long}}} q^{-\frac{1}{2l}(\mathrm{Re}(c_1+\cdots+c_l-\frac{l}{2}+l\langle\alpha,z\rangle))^2} & \text{if } s\neq 0,\ l\neq 0. \end{cases}$$

From (38) and (48), it follows that

$$\begin{aligned} &|\Psi_R(\{b_i-N\},\{c_j-N\};z+\chi)| = |F(\chi;N)G(\chi)^{-1}| \\ &< C_3C_5 \Bigg| \prod_{\substack{\alpha\in B_\chi^+\\ \alpha:\text{short}}} q^{\frac{s}{2}\langle\alpha,\chi\rangle^2+(b_1+\cdots+b_s-\frac{s}{2}+s\langle\alpha,z\rangle)\langle\alpha,\chi\rangle} \\ &\quad\times q^{-\frac{s}{2}(\langle\alpha,\chi\rangle-N)^2-(b_1+\cdots+b_s+s\langle\alpha,z\rangle-\frac{s}{2})(\langle\alpha,\chi\rangle-N)} \\ &\quad\times \prod_{\substack{\alpha\in B_\chi^+\\ \alpha:\text{long}}} q^{\frac{l}{2}\langle\alpha,\chi\rangle^2+(c_1+\cdots+c_l-\frac{l}{2}+l\langle\alpha,z\rangle)\langle\alpha,\chi\rangle} \\ &\quad\times q^{-\frac{l}{2}(\langle\alpha,\chi\rangle-N)^2-(c_1+\cdots+c_l+l\langle\alpha,z\rangle-\frac{l}{2})(\langle\alpha,\chi\rangle-N)} \Bigg| \\ &= C_3C_5 \Bigg| \prod_{\substack{\alpha\in B_\chi^+\\ \alpha:\text{short}}} q^{\frac{sN\langle\alpha,\chi\rangle}{4}+\frac{sN(\langle\alpha,\chi\rangle-N)}{2}+\left(\frac{s\langle\alpha,\chi\rangle}{4}-\frac{s}{2}+b_1+\cdots+b_s+s\langle\alpha,z\rangle\right)N} \\ &\quad\times \prod_{\substack{\alpha\in B_\chi^+\\ \alpha:\text{long}}} q^{\frac{lN\langle\alpha,\chi\rangle}{4}+\frac{lN(\langle\alpha,\chi\rangle-N)}{2}+(\frac{l\langle\alpha,\chi\rangle}{4}-\frac{l}{2}+c_1+\cdots+c_l+l\langle\alpha,z\rangle)N} \Bigg|. \end{aligned} \tag{49}$$

Let N be a large integer satisfying

$$\frac{sN}{4}-\frac{s}{2}+\mathrm{Re}\left(s\langle\alpha,z\rangle+\sum_{i=1}^{s}b_i\right)\geqslant 0 \quad \text{and}$$
$$\frac{lN}{4}-\frac{l}{2}+\mathrm{Re}\left(l\langle\alpha,z\rangle+\sum_{j=1}^{l}c_j\right)\geqslant 0,$$

for $\alpha\in R^+$. For $\alpha\in B_\chi^+$, if α are short and long, then we have

$$\left|q^{\frac{sN(\langle\alpha,\chi\rangle-N)}{2}+\left(\frac{s\langle\alpha,\chi\rangle}{4}-\frac{s}{2}+b_1+\cdots+b_s+s\langle\alpha,z\rangle\right)N}\right|\leqslant 1,$$

and

$$|q^{\frac{lN(\langle\alpha,\chi\rangle-N)}{2}+\left(\frac{l\langle\alpha,\chi\rangle}{4}-\frac{l}{2}+c_1+\cdots+c_l+l\langle\alpha,z\rangle\right)N}|\leqslant 1$$

respectively. Therefore, from (49) and (35), we obtain

$$\begin{aligned}|\Psi_R(\{b_i-N\},\{c_j-N\};z+\chi)| &< C_3C_5\prod_{\substack{\alpha\in B_\chi^+\\ \alpha:\text{short}}}q^{\frac{sN\langle\alpha,\chi\rangle}{4}}\prod_{\substack{\alpha\in B_\chi^+\\ \alpha:\text{long}}}q^{\frac{lN\langle\alpha,\chi\rangle}{4}}\\ &< C_3C_5\,q^{\frac{N}{4}\langle\tilde{\alpha},\chi\rangle},\end{aligned}$$

which completes the proof. □

Lemma 10. *Let s be a non-negative integer. Let H_s be the set of dominant coweights lying on the hyperplane defined by $\langle\tilde{\alpha},\chi\rangle=s+1$, i.e., $H_s:=\{\chi\in D;\langle\tilde{\alpha},\chi\rangle=s+1\}$. Then*

$$\#H_s\leqslant\frac{(s+1)(s+2)\cdots(s+n-1)}{(n-1)!}.$$

Proof. By definition, the highest root $\tilde{\alpha}$ can be written $\tilde{\alpha}=p_1\alpha_1+\cdots+p_n\alpha_n$ where p_i, $i=1,2,\ldots,n$, are non-zero positive integers. For $\chi=v_1\chi_1+\cdots+v_n\chi_n\in D$, the condition $\langle\tilde{\alpha},\chi\rangle=s+1$ is equivalent to $p_1v_1+\cdots+p_nv_n=s+1$. Therefore, we have

$$\#H_s=\#\{(v_1,\ldots,v_n)\in(\mathbf{Z}_{\geqslant 0})^n;\ p_1v_1+\cdots+p_nv_n=s+1\}.$$

Since integers p_i are all positive, we have

$$\begin{aligned}&\#\{(v_1,\ldots,v_n)\in(\mathbf{Z}_{\geqslant 0})^n;\ p_1v_1+\cdots+p_nv_n=s+1\}\\ &\quad\leqslant\#\{(v_1,\ldots,v_n)\in(\mathbf{Z}_{\geqslant 0})^n;\ v_1+\cdots+v_n=s+1\}.\end{aligned}\tag{50}$$

Counting up the integer points $(v_1, \dots, v_n) \in (\mathbf{Z}_{\geqslant 0})^n$ satisfying $v_1 + \cdots + v_n = s+1$ by induction on n, the RHS of (50) is equal to

$$\frac{(s+1)(s+2)\cdots(s+n-1)}{(n-1)!}.$$

This completes the proof. □

Lemma 11. *Let N be a sufficiently large integer. The following holds for N:*

$$\sum_{\chi \in P-D_N} \Psi_R(\{b_i - N\}, \{c_j - N\}; z+\chi) = o\left(q^{\frac{N^2}{4}}\right) \quad (N \to +\infty).$$

Proof. Since P and D_N is W-stable, it follows that

$$P - D_N = \bigcup_{w \in W} w(D - D_N). \tag{51}$$

From (51), it follows that

$$\begin{aligned}
&\sum_{\chi \in P-D_N} \Psi_R(\{b_i - N\}, \{c_j - N\}; z+\chi) \\
&\quad = \sum_{w \in W} \sum_{\chi \in w(D-D_N)} \Psi_R(\{b_i - N\}, \{c_j - N\}; z+\chi) \\
&\quad = \sum_{w \in W} \sum_{\chi \in D-D_N} \Psi_R(\{b_i - N\}, \{c_j - N\}; wz+\chi).
\end{aligned}$$

Thus, it is sufficient to prove that

$$\sum_{\chi \in D-D_N} \Psi_R(\{b_i - N\}, \{c_j - N\}; z+\chi) = o\left(q^{\frac{N^2}{4}}\right) \quad (N \to +\infty). \tag{52}$$

From Lemma 9, it follows that

$$\begin{aligned}
&\left| \sum_{\chi \in D-D_N} \Psi_R(\{b_i - N\}, \{c_j - N\}; z+\chi) \right| \\
&\quad \leqslant \sum_{\chi \in D-D_N} |\Psi_R(\{b_i - N\}, \{c_j - N\}; z+\chi)| < C \sum_{\chi \in D-D_N} q^{\frac{N}{4}\langle \tilde{\alpha}, \chi \rangle}.
\end{aligned} \tag{53}$$

By definition (34), $D - D_N$ is decomposed into H_s's as follows:

$$D - D_N = \bigcup_{m=0}^{\infty} H_{m+N}.$$

From Lemma 10, it follows that

$$\sum_{\chi\in D-D_N} q^{\frac{N}{4}\langle\tilde{\alpha},\chi\rangle} = \sum_{m=0}^{\infty}\sum_{\chi\in H_{m+N}} q^{\frac{N}{4}\langle\tilde{\alpha},\chi\rangle} = \sum_{m=0}^{\infty}(\#H_{m+N})q^{\frac{N}{4}(N+m+1)}$$
$$\leqslant \sum_{m=0}^{\infty}\frac{(m+N+1)(m+N+2)\cdots(m+N+n-1)}{(n-1)!}q^{\frac{N(m+N+1)}{4}}$$
$$=\frac{(N+1)(N+2)\cdots(N+n-1)}{(n-1)!}q^{\frac{N(N+1)}{4}}+\cdots,$$

so that

$$\left(\sum_{\chi\in D-D_N} q^{\frac{N}{4}\langle\tilde{\alpha},\chi\rangle}\right)\Bigg/ q^{\frac{N^2}{4}} \leqslant \frac{N^{n-1}q^{\frac{N}{4}}}{(n-1)!}+\cdots \to 0 \quad (N\to+\infty). \tag{54}$$

From (53) and (54) it follows (52), completing the proof of Lemma 11. □

We now prove Theorem 6.

Proof of Theorem 6. By using Lemma 11, we have

$$\begin{aligned} &M_R(\{b_i-N\},\{c_j-N\};P;z)\\ &\quad=\sum_{\chi\in D_N}\Psi_R(\{b_i-N\},\{c_j-N\};z+\chi)\\ &\qquad+\sum_{\chi\in P-D_N}\Psi_R(\{b_i-N\},\{c_j-N\};z+\chi)\\ &\quad=\sum_{\chi\in D_N}\Psi_R(\{b_i-N\},\{c_j-N\};z+\chi)+o\left(q^{\frac{N^2}{4}}\right)\quad(N\to+\infty). \end{aligned} \tag{55}$$

From Lemma 8 and (55), we obtain

$$\lim_{N\to+\infty} M_R(\{b_i-N\},\{c_j-N\};P;z) = M_R(P;z). \tag{56}$$

In particular, when $(s,l)=(1,1)$ it follows that

$$\lim_{N\to+\infty} C_R(b_1-N,c_1-N;P) = \lim_{N\to+\infty} M_R(b_1-N,c_1-N;P;z) = M_R(P;z).$$

On the other hand, from Proposition 3, we have

$$\lim_{N\to+\infty} C_R(b_1-N,c_1-N;P) = |P/Q|(q)_\infty^n.$$

Thus we have established

$$M_R(P;z) = |P/Q|(q)_\infty^n. \tag{57}$$

By using (20), this argument is valid for $M_R(Q;z)=(q)_\infty^n$. The proof of Theorem 6 is now complete. □

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Appendix A. Gustafson's G_2-type summation formula

We consider the sum $J_{G_2}(b_1,b_2,b_3,b_4;P;z)$ of the case $(s,l)=(4,0)$ for G_2-type root system. The aim of this section is to give another proof of the following theorem established by Gustafson [5], as an application of Theorem 6:

Theorem A.1 (Gustafson). *If* $q<|q^{b_1+b_2+b_3+b_4}|^2$, *the sum* $J_{G_2}(b_1,b_2,b_3,b_4;P;z)$ *converges and is expressed in the form* (9). *The constant* $C_{G_2}(b_1,b_2,b_3,b_4;P)$ *is expressed as*

$$\frac{(q)_\infty^2(q^{1-b_1-b_2-b_3-b_4})_\infty}{(q^{1-2(b_1+b_2+b_3+b_4)})_\infty}\prod_{i=1}^{4}\frac{(q^{1-2b_i})_\infty}{(q^{1-b_i})_\infty}\prod_{1\leqslant i<j\leqslant 4}(q^{1-b_i-b_j})_\infty$$
$$\times\prod_{1\leqslant i<j<k\leqslant 4}(q^{1-b_i-b_j-b_k})_\infty.$$

The former part of Theorem A.1 was mentioned in (16). Before proving the theorem, we give a lemma in the next section. By using notation in Examples, we write the sum $J_{G_2}(b_1,b_2,b_3,b_4;P;z)$ explicitly as

$$J_{G_2}(b_1,b_2,b_3,b_4;P;z)=\sum_{\chi\in P}\Phi_{G_2}(b_1,b_2,b_3,b_4;z+\chi)\Delta_{G_2}(z+\chi), \tag{A.1}$$

where

$$\begin{aligned}\Phi_{G_2}&(b_1,b_2,b_3,b_4;x)\\&=\prod_{i=1}^{4}q^{(1-2b_i)\langle 2\alpha_1+\alpha_2,x\rangle}\\&\quad\times\frac{(q^{1-b_i+\langle\alpha_1,x\rangle})_\infty}{(q^{b_i+\langle\alpha_1,x\rangle})_\infty}\frac{(q^{1-b_i+\langle\alpha_1+\alpha_2,x\rangle})_\infty}{(q^{b_i+\langle\alpha_1+\alpha_2,x\rangle})_\infty}\frac{(q^{1-b_i+\langle 2\alpha_1+\alpha_2,x\rangle})_\infty}{(q^{b_i+\langle 2\alpha_1+\alpha_2,x\rangle})_\infty}\end{aligned} \tag{A.2}$$

$$\begin{aligned}\Delta_{G_2}(x)=&\left(q^{\frac12\langle\alpha_1,x\rangle}-q^{-\frac12\langle\alpha_1,x\rangle}\right)\left(q^{\frac12\langle\alpha_1+\alpha_2,x\rangle}-q^{-\frac12\langle\alpha_1+\alpha_2,x\rangle}\right)\\&\times\left(q^{\frac12\langle 2\alpha_1+\alpha_2,x\rangle}-q^{-\frac12\langle 2\alpha_1+\alpha_2,x\rangle}\right)\left(q^{\frac12\langle\alpha_2,x\rangle}-q^{-\frac12\langle\alpha_2,x\rangle}\right)\\&\times\left(q^{\frac12\langle 3\alpha_1+\alpha_2,x\rangle}-q^{-\frac12\langle 3\alpha_1+\alpha_2,x\rangle}\right)\left(q^{\frac12\langle 3\alpha_1+2\alpha_2,x\rangle}-q^{-\frac12\langle 3\alpha_1+2\alpha_2,x\rangle}\right)\end{aligned} \tag{A.3}$$

and $P=\mathbf{Z}\chi_1+\mathbf{Z}\chi_2$.

A.1. Recurrence relation of $J_{G_2}(b_1, b_2, b_3, b_4; P; z)$

Lemma A.2. *The recurrence relation of $J_{G_2}(b_1, b_2, b_3, b_4; P; z)$ is the following*:

$$J_{G_2}(b_1+1, b_2, b_3, b_4; \xi) = r_{G_2}(b_1, b_2, b_3, b_4) J_{G_2}(b_1, b_2, b_3, b_4; P; z),$$

where

$$\begin{aligned} & r_{G_2}(b_1, b_2, b_3, b_4) \\ & \quad = -(1-q^{b_1+b_2})(1-q^{b_1+b_3})(1-q^{b_1+b_4}) \\ & \qquad \times (1-q^{b_1+b_2+b_3})(1-q^{b_1+b_2+b_4})(1-q^{b_1+b_3+b_4}) \\ & \qquad \times \frac{(1-q^{2b_1})(1-q^{2b_1+1})(1-q^{b_1+b_2+b_3+b_4})}{q^{3b_1}(1-q^{b_1})(1-q^{2(b_1+b_2+b_3+b_4)})(1-q^{2(b_1+b_2+b_3+b_4)+1})}. \end{aligned}$$

Proof. We set the half sum of the positive roots and the fundamental weights as

$$\rho := \tfrac{1}{2}\sum_{\alpha>0} \alpha = 5\alpha_1 + 3\alpha_2, \quad \eta_1 := 2\alpha_1 + \alpha_2, \quad \eta_2 := 3\alpha_1 + 2\alpha_2.$$

We denote by w_α the reflection defined by $w_\alpha(x) := x - \langle \alpha^\vee, x \rangle \alpha$. The Weyl group W is generated by w_{α_1} and w_{α_2}, which is isomorphic to the dihedral group of order 12. For a function $f(x)$, we denote by $\mathscr{A}f(x)$ the alternating sum of $f(x)$ with the action of W, i.e.,

$$\mathscr{A}f(x) := \sum_{w \in W} \operatorname{sgn} w \; wf(x) = \sum_{w \in W} \operatorname{sgn} w \, f(w^{-1}x).$$

In particular, we use

$$\mathscr{A}_\lambda(x) := \mathscr{A}(q^{\langle \lambda, x \rangle}) = \sum_{w \in W} \operatorname{sgn} w \; q^{\langle w\lambda, x \rangle}.$$

The Weyl denominator formula says that

$$\mathscr{A}_\rho(x) = \Delta_{G_2}(x). \tag{A.4}$$

From (A.2), it follows that

$$\begin{aligned} & \frac{\Phi_{G_2}(b_1+1, b_2, b_3, b_4; x)}{\Phi_{G_2}(b_1, b_2, b_3, b_4; x)} \\ & \quad = S_3(x) - (q^{b_1} + q^{-b_1}) S_2(x) + (q^{b_1} + q^{-b_1})^2 S_1(x) - (q^{b_1} + q^{-b_1})^3, \end{aligned} \tag{A.5}$$

where

$$\begin{aligned} S_1(x) = {} & (q^{\langle \alpha_1, x \rangle} + q^{-\langle \alpha_1, x \rangle}) + (q^{\langle \alpha_1+\alpha_2, x \rangle} + q^{-\langle \alpha_1+\alpha_2, x \rangle}) \\ & + (q^{\langle 2\alpha_1+\alpha_2, x \rangle} + q^{-\langle 2\alpha_1+\alpha_2, x \rangle}), \end{aligned}$$

$$S_2(x) = (q^{\langle \alpha_1,x\rangle} + q^{-\langle \alpha_1,x\rangle})(q^{\langle 2\alpha_1+\alpha_2,x\rangle} + q^{-\langle 2\alpha_1+\alpha_2,x\rangle}) + (q^{\langle \alpha_1,x\rangle} + q^{-\langle \alpha_1,x\rangle})(q^{\langle \alpha_1+\alpha_2,x\rangle} + q^{-\langle \alpha_1+\alpha_2,x\rangle}) + (q^{\langle \alpha_1+\alpha_2,x\rangle} + q^{-\langle \alpha_1+\alpha_2,x\rangle})(q^{\langle 2\alpha_1+\alpha_2,x\rangle} + q^{-\langle 2\alpha_1+\alpha_2,x\rangle}),$$

$$S_3(x) = (q^{\langle \alpha_1,x\rangle} + q^{-\langle \alpha_1,x\rangle})(q^{\langle \alpha_1+\alpha_2,x\rangle} + q^{-\langle \alpha_1+\alpha_2,x\rangle}) \times (q^{\langle 2\alpha_1+\alpha_2,x\rangle} + q^{-\langle 2\alpha_1+\alpha_2,x\rangle}).$$

Each $S_j(x)$ satisfies the following equations:

$$S_1(x)\mathscr{A}_\rho(x) = \mathscr{A}_{\rho+\eta_1}(x) - \mathscr{A}_\rho(x), \tag{A.6}$$

$$S_2(x)\mathscr{A}_\rho(x) = \mathscr{A}_{\rho+\eta_2}(x) - 2\mathscr{A}_\rho(x), \tag{A.7}$$

$$S_3(x)\mathscr{A}_\rho(x) = \mathscr{A}_{\rho+2\eta_1}(x) - \mathscr{A}_{\rho+\eta_2}(x) - \mathscr{A}_{\rho+\eta_1}(x) + 2\mathscr{A}_\rho(x). \tag{A.8}$$

For simplicity we abbreviate $\Phi_{G_2}(b_1,b_2,b_3,b_4;x)$ to $\Phi_{G_2}(x)$. We define $J_\eta(z)$ by

$$J_\eta(z) := \sum_{\chi\in P} \Phi_{G_2}(z+\chi)\mathscr{A}_{\rho+\eta}(z+\chi). \tag{A.9}$$

By this definition and (A.4), it is obvious that $J_0(z) = J_{G_2}(b_1,b_2,b_3,b_4;P;z)$. From (A.5)–(A.8), we get

$$\begin{aligned} &J_{G_2}(b_1+1,b_2,b_3,b_4;P;z) \\ &\quad = (J_{2\eta_1}(z) - J_{\eta_2}(z) - J_{\eta_1}(z) + 2J_0(z)) - (q^{b_1}+q^{-b_1})(J_{\eta_2}(z) - 2J_0(z)) \\ &\qquad + (q^{b_1}+q^{-b_1})^2(J_{\eta_1}(z) - J_0(z)) - (q^{b_1}+q^{-b_1})^3 J_0(z). \end{aligned} \tag{A.10}$$

For a function $\varphi(x)$, we define $\nabla_\chi\varphi(x)$ as

$$\nabla_\chi\varphi(x) := \varphi(x) - \frac{\Phi_{G_2}(x+\chi)}{\Phi_{G_2}(x)}\varphi(x+\chi) \quad \text{for } \chi\in P.$$

Then, we obtain

$$\sum_{\lambda\in P} \Phi_{G_2}(z+\lambda)\nabla_\chi\varphi(z+\lambda) = 0, \tag{A.11}$$

because the sum $\sum_{\lambda\in P}\Phi_{G_2}(z+\lambda)\varphi(z+\lambda)$ defined over the lattice P is invariant under the shift $z\to z+\chi$ for $\chi\in P$. Equation (A.11) implies that

$$\sum_{\lambda\in P} \Phi_{G_2}(z+\lambda)\mathscr{A}\nabla_\chi\varphi(z+\lambda) = 0. \tag{A.12}$$

In particular, for the fundamental coweight $\chi_2\in P$, it follows that

$$\frac{\Phi_{G_2}(x+\chi_2)}{\Phi_{G_2}(x)} = q^{4-2(b_1+b_2+b_3+b_4)} \prod_{i=1}^{4} \frac{(1-q^{b_i+\langle \alpha_1+\alpha_2,x\rangle})(1-q^{b_i+\langle 2\alpha_1+\alpha_2,x\rangle})}{(1-q^{1-b_i+\langle \alpha_1+\alpha_2,x\rangle})(1-q^{1-b_i+\langle 2\alpha_1+\alpha_2,x\rangle})}.$$

Moreover, for $\nabla_{\chi_2}\varphi(x)$, we now take $\varphi(x)$ as

$$\varphi(x) = q^{\langle m_1\alpha_1+m_2\alpha_2,x\rangle+2(b_1+b_2+b_3+b_4)} \prod_{i=1}^{4} (1-q^{-b_i+\langle \alpha_1+\alpha_2,x\rangle})(1-q^{-b_i+\langle 2\alpha_1+\alpha_2,x\rangle})$$

of the cases $(m_1,m_2)=(-5,-4),(-3,-3)$ and $(-3,-4)$. Then, after some direct calculation of $\mathscr{A}\nabla_{\chi_2}\varphi(x)$ for $\varphi(x)$ above, by using (A.9) and (A.12), we obtain the following three equations:

$$0=(1+B_4)J_{\eta_1}(z)-(B_1+B_2+B_3)J_0(z), \tag{A.13}$$

$$\begin{aligned}0=&(1-qB_4^2)J_{2\eta_1}(z)+[B_4-B_1-B_2+qB_4(B_3+B_2-1)]J_{\eta_1}(z)\\&+[B_2+B_1B_2+B_1B_3-B_2B_4-B_3B_4\\&-q(B_1+B_2-B_1B_3-B_2B_3-B_2B_4)]J_0(z),\end{aligned} \tag{A.14}$$

$$\begin{aligned}0=&(B_1B_4+B_4^2-1-B_3)J_{2\eta_1}(z)+(B_1+B_2-B_2B_3-B_2B_4)J_{\eta_2}(z)\\&+(B_1-B_1B_2+B_2B_3-B_3B_4)J_{\eta_1}(z)\\&+B_2(B_3+B_4-1-B_1)J_0(z),\end{aligned} \tag{A.15}$$

where B_j is the jth elementary symmetric polynomial of q^{b_i}, i.e., $B_1=q^{b_1}+q^{b_2}+q^{b_3}+q^{b_4}$, $B_2=q^{b_1+b_2}+q^{b_1+b_3}+q^{b_1+b_4}+q^{b_2+b_3}+q^{b_2+b_4}+q^{b_3+b_4}$, $B_3=q^{b_1+b_2+b_3}+q^{b_1+b_2+b_4}+q^{b_1+b_3+b_4}+q^{b_2+b_3+b_4}$, $B_4=q^{b_1+b_2+b_3+b_4}$. By eliminating $J_{2\eta_1}(z)$, $J_{\eta_2}(z)$ and $J_{\eta_1}(z)$ from Eqs. (A.10), (A.13)–(A.15), we eventually obtain the recurrence relation in Lemma A.2. □

A.2. Application of Theorem 6

Proof of Theorem A.1. From (19), Lemma A.2 and the recurrence relation of $\Theta_{G_2}(b_1,b_2,b_3,b_4;P;z)$

$$\Theta_{G_2}(b_1+1,b_2,b_3,b_4;P;z)=-q^{3b_1}\Theta_{G_2}(b_1,b_2,b_3,b_4;P;z), \tag{A.16}$$

it follows that

$$\begin{aligned}&C_{G_2}(b_1,b_2,b_3,b_4;P;z)\\&=\frac{J_{G_2}(b_1,b_2,b_3,b_4;P;z)}{\Theta_{G_2}(b_1,b_2,b_3,b_4;P;z)}\\&=\prod_{i=1}^{4}\frac{(q^{1-2b_i})_{2N}}{(q^{1-b_i})_N}\prod_{1\leqslant i<j\leqslant 4}(q^{1-b_i-b_j})_{2N}\prod_{1\leqslant i<j<k\leqslant 4}(q^{1-b_i-b_j-b_k})_{3N}\\&\quad\times\frac{(q^{1-b_1-b_2-b_3-b_4})_{4N}}{(q^{1-2(b_1+b_2+b_3+b_4)})_{8N}}\frac{J_R(b_1-N,b_2-N,b_3-N,b_4-N;P;z)}{\Theta_R(b_1-N,b_2-N,b_3-N,b_4-N;P;z)}\end{aligned}$$

$$= \prod_{i=1}^{4} \frac{(q^{1-2b_i})_{2N}}{(q^{1-b_i})_N} \prod_{1\leqslant i<j\leqslant 4} (q^{1-b_i-b_j})_{2N} \prod_{1\leqslant i<j<k\leqslant 4} (q^{1-b_i-b_j-b_k})_{3N}$$

$$\times \frac{(q^{1-b_1-b_2-b_3-b_4})_{4N}}{(q^{1-2(b_1+b_2+b_3+b_4)})_{8N}} M_{G_2}(b_1-N, b_2-N, b_3-N, b_4-N; P; z)$$

$$= \frac{(q^{1-b_1-b_2-b_3-b_4})_\infty}{(q^{1-2(b_1+b_2+b_3+b_4)})_\infty} \prod_{i=1}^{4} \frac{(q^{1-2b_i})_\infty}{(q^{1-b_i})_\infty}$$

$$\times \prod_{1\leqslant i<j\leqslant 4} (q^{1-b_i-b_j})_\infty \prod_{1\leqslant i<j<k\leqslant 4} (q^{1-b_i-b_j-b_k})_\infty$$

$$\times \lim_{N\to+\infty} M_{G_2}(b_1-N, b_2-N, b_3-N, b_4-N; P; z). \tag{A.17}$$

Combining (A.17) and Theorem 6, we obtain Theorem A.1. □

Remark. For the constant $C_{F_4}(b_1, b_2, b_3; P)$ of the case $(s, l) = (3, 0)$ for F_4-type root system, we have the following theorem to do the same process as above:

Theorem A.3. *If $q < |q^{b_1+b_2+b_3}|^6$, the sum $J_{F_4}(b_1, b_2, b_3; P; z)$ converges and is expressed in form* (9). *The constant $C_{F_4}(b_1, b_2, b_3; P)$ is expressed as follows*:

$$C_{F_4}(b_1, b_2, b_3; P)$$

$$= (q)_\infty^4 (q^{1-b_1-b_2})_\infty (q^{1-b_2-b_3})_\infty (q^{1-b_1-b_3})_\infty$$

$$\times (q^{1-b_1-2b_2})_\infty (q^{1-b_1-2b_3})_\infty (q^{1-b_2-2b_1})_\infty$$

$$\times (q^{1-b_2-2b_3})_\infty (q^{1-b_3-2b_1})_\infty (q^{1-b_3-2b_2})_\infty$$

$$\times (q^{1-b_1-b_2-b_3})_\infty (q^{1-b_1-b_2-2b_3})_\infty (q^{1-b_1-2b_2-b_3})_\infty (q^{1-2b_1-b_2-b_3})_\infty$$

$$\times (q^{1-b_1-2b_2-2b_3})_\infty (q^{1-2b_1-b_2-2b_3})_\infty (q^{1-2b_1-2b_2-b_3})_\infty (q^{1-2b_1-2b_2-2b_3})_\infty$$

$$\times \frac{(q^{1-3b_1-3b_2-3b_3})_\infty}{(q^{1-6b_1-6b_2-6b_3})_\infty} \prod_{i=1}^{3} \frac{(q^{1-2b_i})_\infty}{(q^{1-b_i})_\infty} \frac{(q^{1-3b_i})_\infty}{(q^{1-b_i})_\infty}.$$

Proof. The former part was mentioned in (17). For evaluation of the constant $C_{F_4}(b_1, b_2, b_3; P)$, see [9]. □

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